Non-quadratic solutions to the Monge-Ampère equation

Yifei Pan and Yuan Zhang

Abstract

We construct ample smooth strictly plurisubharmonic non-quadratic solutions to the Monge-Ampère equation on either cylindrical type domains or the whole complex Euclidean space $\mathbb{C}^2$. Among these, the entire solutions defined on $\mathbb{C}^2$ induce flat Kähler metrics, as expected by a question of Calabi. In contrast, those on cylindrical domains produce a family of nowhere flat Kähler metrics. Beyond these smooth solutions, we also classify solutions that are radially symmetric in one variable, which exhibit various types of singularities. Finally, we explore analogous solutions to Donaldson's equation motivated by a result of He.

1 Introduction

According to fundamental works by Jörgens [10] in dimension 2 and by Calabi [4] and Pogorelov [15] in higher dimensions, any entire convex viscosity solution of the real Monge-Ampère equation

$$\det(D^2 u) = 1 \quad \text{on} \quad \mathbb{R}^m$$

must be a quadratic function. See also, for instance, [12]. The similar property no longer holds for entire plurisubharmonic solutions to the complex Monge-Ampère equation in $\mathbb{C}^n$:

$$\det(\partial\bar{\partial} u) = 1. \tag{1.1}$$

Here $\partial\bar{\partial} u$ is the complex Hessian of u. Such solutions give rise to Kähler metrics via the complex Hessian whose associated volume forms are constant. Calabi posted in [5] the question of whether these Kähler metrics are flat, which still remains open.

Motivated by those results and Calabi's question, we investigate solutions to (1.1) in $\mathbb{C}^2$ that depend quadratically on one variable. Specifically, for variables $(z, w) \in \mathbb{C}^2$, we focus on real-valued solutions that are quadratic only in the z direction:

$$u(z, w) = a(w)|z|^2 + b(w)z^2 + \overline{b(w)z^2} + c(w)z + \overline{c(w)z} + d(w), \tag{1.2}$$

where a, b, c and d are smooth functions of w, and $a > 0$. This particular form enables us to convert the fully nonlinear equation (1.1) to a system of simpler semi-linear elliptic equations, whose solutions can be effectively approached. See (2.1)-(2.3).

2020 Mathematics Subject Classification. Primary 32W20; Secondary 32Q15.
Key words and phrases. Monge-Ampère equation, flat metric, radial, explicit examples, Donaldson's equation.

In Section 2, by applying existence theorems from [16] for nonlinear systems of Poisson equations, we construct ample smooth, strictly plurisubharmonic solutions over cylindrical-type domains of the form $\mathbb{C} \times D_R$ for any $R > 0$, where D_R denotes the disc of radius R in $\mathbb{C}$. See Theorem 2.3. Each of these solutions possesses the special quadric form given in (1.2), yet they are not quadratic in the w-variable. Note that all these solutions can be trivially extended to the n dimensional case by adding quadratic terms $|z_3|^2 + \cdots |z_n|^2$.

To fully understand Calabi's question in our setting, we begin in Section 3 by establishing an obstruction that prevents the induced Kähler metric from being flat.

Theorem 1.1. *Let D be a domain in $\mathbb{C}$. Suppose u is a plurisubharmonic solution to* (1.1) *on $\mathbb{C} \times D$ of the form* (1.2)*. Then the corresponding Kähler metric is flat on $\mathbb{C} \times D$ if and only if b is holomorphic on D.*

Making use of this theorem one can construct solutions to (1.1) whose associated Kähler metric is nowhere flat over cylindrical-type domains. See Theorem 3.1 and Example 1. In particular, these examples demonstrate that Calabi's question does not hold on any bounded domains.

In Section 4, we produce in Theorem 4.1 entire solutions of (1.1) of the special form (1.2) on $\mathbb{C}^2$. Note that the Kähler metrics induced by all these solutions are flat, and therefore do not provide counterexamples to Calabi's question. Although it remains unclear whether the question holds in full generality, we show in Theorem 4.2 that for every entire solutions of the form (1.2), $\frac{\partial b}{\partial \bar{w}}$ must have zeros somewhere.

In addition to the aforementioned smooth solutions, we also explore in Section 5 solutions that are radially symmetric on w and exhibit singularities. In fact, we derive explicit expressions for all such solutions. At the end of the section, we present a variety of examples with distinct singular behavior (see Examples 3-6). Notably, these recover several existing known examples in the literature, such as Błocki [2], He [8], and Wang-Wang [18].

In Section 6, more general forms of solutions that are non-quadratic in both variables are discussed, for instance, by replacing z in (1.2) with a holomorphic function ϕ of z. As shown in Theorem 6.1, in order for this to yield a solution to (1.1), the function ϕ must take a very rigid dichotomous form as described in (6.3) after normalization. Solutions involving such ϕ are then constructed in Theorem 6.2.

While investigating the geometric structure for the space of volume forms on compact Riemannian manifolds, Donaldson introduced the operator $u_{tt}\Delta u - |\nabla u_t|^2$, where $(t, x) \in \mathbb{R} \times \mathbb{R}^m$, and the Laplacian Δ and the gradient ∇ are both with respect to the space x variable. When $m = 2$, by complexifying the t direction, Donaldson's operator can be reduced to a special case of the complex Monge-Ampère operator $\det(\partial\bar{\partial}u)$. In Section 7, we generalize a result of He [7] on solutions to

$$u_{tt}\Delta u - |\nabla u_t|^2 = 1, \tag{1.3}$$

and obtain a larger class of solutions on cylindrical domains $\mathbb{R} \times B_R$ that are quadratic in the t variable in Theorem 7.1. Here B_R is the ball of radius R in $\mathbb{R}^m$. However, for entire solutions, Theorem 7.2 shows that every entire solution of the form (7.2) must reduce to He's original case.

2 Solvability on cylindrical domains

We first derive conditions for the coefficients a, b, c and d in (1.2) so that they yield solutions to (1.1). Since u is real-valued, so are a and d. A straightforward computation gives the complex

Hessian of u below.

$$\partial\bar{\partial}u := \begin{bmatrix} u_{\bar{z}z} & u_{z\bar{w}} \\ u_{\bar{z}w} & u_{\bar{w}w} \end{bmatrix} = \begin{bmatrix} a & a_{\bar{w}}\bar{z} + 2b_{\bar{w}}z + c_{\bar{w}} \\ a_w z + 2\bar{b}_w\bar{z} + \bar{c}_w & a_{\bar{w}w}|z|^2 + b_{\bar{w}w}z^2 + \overline{b_{\bar{w}w}z^2} + c_{\bar{w}w}z + \overline{c_{\bar{w}w}z} + d_{\bar{w}w} \end{bmatrix}.$$

Here $f_w := \frac{\partial f}{\partial w}$ for a smooth function f, and similarly $f_{\bar{w}} := \frac{\partial f}{\partial \bar{w}}$. Consequently, taking the determinant of the Hessian, and sorting it out according to the powers of z and $\bar{z}$, we obtain

$$\begin{aligned} \det(\partial\bar{\partial}u) =& |z|^2\left(aa_{\bar{w}w} - |a_w|^2 - 4|b_{\bar{w}}|^2\right) + z^2\left(ab_{\bar{w}w} - 2a_w b_{\bar{w}}\right) + \bar{z}^2\left(a\overline{b_{\bar{w}w}} - 2a_{\bar{w}}\bar{b}_w\right) \\ &+ z\left(ac_{\bar{w}w} - a_w c_{\bar{w}} - 2b_{\bar{w}}\bar{c}_w\right) + \bar{z}\left(a\bar{c}_{\bar{w}w} - a_{\bar{w}}\bar{c}_w - 2\bar{b}_w c_{\bar{w}}\right) + \left(ad_{\bar{w}w} - |c_{\bar{w}}|^2\right). \end{aligned}$$

Thus any solution u to (1.1) of the form (1.2) should satisfy the following system of semilinear differential equations:

$$\begin{cases} aa_{\bar{w}w} = |a_w|^2 + 4|b_{\bar{w}}|^2; \\ ab_{\bar{w}w} = 2a_w b_{\bar{w}}; \\ ac_{\bar{w}w} = a_w c_{\bar{w}} + 2b_{\bar{w}}\bar{c}_w; \\ ad_{\bar{w}w} = |c_{\bar{w}}|^2 + 1. \end{cases} \tag{2.1}$$

To further simplify (2.1), noting that $a > 0$, let

$$\tilde{a} := \ln a, \tag{2.2}$$

or equivalently, $a = e^{\tilde{a}}$. Then $a_w = e^{\tilde{a}}\tilde{a}_w$ and $a_{\bar{w}w} = e^{\tilde{a}}\left(\tilde{a}_{\bar{w}w} + |\tilde{a}_w|^2\right)$. This transformation allows us to rewrite the system to be

$$\begin{cases} \tilde{a}_{\bar{w}w} = 4e^{-2\tilde{a}}|b_{\bar{w}}|^2; \\ b_{\bar{w}w} = 2\tilde{a}_w b_{\bar{w}}; \\ c_{\bar{w}w} = \tilde{a}_w c_{\bar{w}} + 2e^{-\tilde{a}}b_{\bar{w}}\bar{c}_w; \\ d_{\bar{w}w} = e^{-\tilde{a}}\left(|c_{\bar{w}}|^2 + 1\right). \end{cases} \tag{2.3}$$

Namely, any solution to (2.3) on a domain $D \subset \mathbb{C}$ leads to a solution u to (1.1) on $\mathbb{C} \times D$.

To show the local existence of solutions to (2.3), one can make use of the following existence theorem for general nonlinear systems of Poisson equations in [16].

Theorem 2.1. *[16, Theorem 1.4] Let $F = (F_1, \dots, F_N)$ be a $C^{1,\alpha}$ vector-valued function in $\mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^{mN}$ for some $0 < \alpha < 1$. Given any initial jets $(c_0, c_1) \in \mathbb{R}^N \times \mathbb{R}^{mN}$, there exist infinitely many $C^{2,\alpha}$ solutions $v = (v_1, \dots, v_N)$ satisfying*

$$\begin{cases} \Delta v = F(\cdot, v, \nabla v); \\ v(0) = c_0; \\ \nabla v(0) = c_1 \end{cases}$$

in some small neighborhood of 0 in $\mathbb{R}^m$.

This gives ample smooth solutions for the pair $(\tilde{a}, b)$ in (2.3) near a neighborhood of 0, whose jets up to order 1 at 0 can be prescribed arbitrarily. Consequently, a rescaling method can be used to obtain solutions on any bounded domain. This is due to the special structure of the equation (1.1), and the fact that if u is a solution on B_r for some $r > 0$, then $\tilde{u}(z, w) := \frac{R^2}{r^2}u(rz/R, rw/R)$ is a solution on B_R for any $R > 0$.

Alternatively we present another approach to obtaining infinitely many solutions to (2.3) that do not rely on the rescaling process, but instead by making use of an existence theorem [16, Theorem 1.6] on large domains as follows.

Theorem 2.2. *[16, Theorem 1.6] Let $F = (F_1, \ldots, F_N)$ be a C^2 vector-valued function of variables $(X, Y) \in \mathbb{R}^N \times \mathbb{R}^{mN}$, with $F(0) = 0$ and $\nabla F(0) = 0$. For any $R > 0$ and $0 < \alpha < 1$, there exist infinitely many $C^{2,\alpha}$ solutions $v = (v_1, \ldots, v_N)$ to the partial differential system*

$$\Delta v = F(v, \nabla v) \quad on \quad B_R.$$

Moreover, all these solutions are of vanishing order 2 at $0 \in \mathbb{R}^m$. Namely. $v(0) = 0, \nabla v(0) = 0, \nabla^2 v(0) \neq 0$.

It should be pointed out that, although [16] only focuses on real-valued systems, by splitting v and F into real and imaginary parts accordingly, Theorem 2.1 and Theorem 2.2 can be easily applied to complex-valued systems that we will be having.

Theorem 2.3. *For each $R > 0$, there exist infinitely many smooth strictly plurisubharmonic functions that satisfy (1.1) on $\mathbb{C} \times D_R$. All of these solutions take on a special quadric form (1.2) in the first variable z.*

Proof. Take $v = (\tilde{a}, b)$ in Theorem 2.2, and $F(X_1, X_2, Y_1, Y_2, Y_3, Y_4) = (16e^{-2X_1}|Y_4|^2, 8Y_1Y_4)$. One can check that

$$\Delta(\tilde{a}, b) = F(\tilde{a}, b, \tilde{a}_w, b_w, \tilde{a}_{\bar{w}}, b_{\bar{w}}),$$

with $F(0) = 0$ and $\nabla F(0) = 0$. Theorem 2.2 thus gives rise to infinitely many $C^{2,\alpha}$ solutions $(\tilde{a}, b)$ on D_R which are of vanishing order 2 at 0. By a standard bootstrapping argument, these solutions are necessarily smooth on D_R.

Now, with each such pair of solution $(\tilde{a}, b)$, let c be any holomorphic function. Then c automatically satisfies

$$c_{\bar{w}w} = \tilde{a}_w c_{\bar{w}} + 2e^{-\tilde{a}} b_{\bar{w}} \bar{c}_w \quad \text{on} \quad D_R. \tag{2.4}$$

Finally, substituting such $(\tilde{a}, b, c)$ into the linear equation:

$$d_{\bar{w}w} = e^{-\tilde{a}} \left(|c_{\bar{w}}|^2 + 1\right) \quad \text{on} \quad D_R \tag{2.5}$$

to solve for a smooth d on D_R. Altogether, we have obtained a, b, c and d such that the system (2.1) is satisfied, and thus u of the form (1.2) with these coefficients provides infinitely many solutions to (1.1). That the solutions are strictly plurisubharmonic is because we are dealing with 2×2 Hessians. □

In particular, the theorem generates infinitely many smooth Kähler metrics $\partial\bar{\partial}u$ whose volume forms are constant. In fact, for any smooth positive function f on $\mathbb{C} \times D_R$, the similar approach as in the proof can be applied to produce infinitely many smooth strictly plurisubharmonic functions defined on $\mathbb{C} \times D_R$ that satisfy

$$\det(\partial\bar{\partial}u) = f \quad \text{on} \quad \mathbb{C} \times D_R,$$

3 An obstruction to a flat metric

In this section we prove Theorem 1.1: an obstruction for a solution to (1.1) of the form (1.2) to induce a flat Kähler metric. As an application, this leads to the construction of a large class of solutions, not obtained via rescaling, to (1.1) such that the associated Kähler metrics are nowhere flat on cylindrical domain $\mathbb{C} \times D_R$ for any $R > 0$.

Proof of Theorem 1.1: We first compute the Kähler metric $g = \partial\bar\partial u$ for a solution u of the form (1.2).

$$g = \begin{bmatrix} g_{1\bar 1} & g_{1\bar 2} \\ g_{2\bar 1} & g_{2\bar 2} \end{bmatrix} := \begin{bmatrix} a & a_{\bar w}\bar z + 2b_{\bar w}z + c_{\bar w} \\ a_w z + 2\bar b_w \bar z + \bar c_w & a_{\bar w w}|z|^2 + b_{\bar w w}z^2 + \overline{b_{\bar w w}z^2} + c_{\bar w w}z + \overline{c_{\bar w w}z} + d_{\bar w w} \end{bmatrix}. \tag{3.1}$$

Since $\det g = 1$, the inverse matrix g^{-1} of g is

$$g^{-1} = \begin{bmatrix} g^{\bar 1 1} & g^{\bar 1 2} \\ g^{\bar 2 1} & g^{\bar 2 2} \end{bmatrix} = \begin{bmatrix} a_{\bar w w}|z|^2 + b_{\bar w w}z^2 + \overline{b_{\bar w w}z^2} + c_{\bar w w}z + \overline{c_{\bar w w}z} + d_{\bar w w} & -a_{\bar w}\bar z - 2b_{\bar w}z - c_{\bar w} \\ -a_w z - 2\bar b_w \bar z - \bar c_w & a \end{bmatrix}.$$

According to the general formulae for Christoffel symbols under a Kähler metric g (see, for instance, [1] [11]):

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{\partial g_{\gamma\bar\nu}}{\partial z_\beta} g^{\bar\nu\alpha}, \alpha, \beta, \gamma \in \{1, \dots, n\}, \tag{3.2}$$

a direct computation gives

$$\Gamma^1_{11} = \frac{\partial g_{1\bar 1}}{\partial z} g^{\bar 1 1} + \frac{\partial g_{1\bar 2}}{\partial z} g^{\bar 2 1} = -\,2b_{\bar w}\left(a_w z + 2\bar b_w \bar z + \bar c_w\right).$$

Since the Kähler metric is flat, the curvature tensor $R \equiv 0$. In particular,

$$R^1_{1\bar 1 1} = -\frac{\partial \Gamma^1_{11}}{\partial \bar z} = 4|b_{\bar w}|^2 \equiv 0. \tag{3.3}$$

Namely, b must be holomorphic on D.

Conversely, if $b_{\bar w} \equiv 0$ on D, then by (2.3)

$$\begin{cases} \tilde a_{\bar w w} = 0; \\ c_{\bar w w} = \tilde a_w c_{\bar w}; \\ d_{\bar w w} = e^{-\tilde a}\left(|c_{\bar w}|^2 + 1\right). \end{cases} \tag{3.4}$$

From the first equation above, there exists some holomorphic function h on D such that

$$\tilde a = 2\mathrm{Re}(h), \quad \text{and so} \quad a = e^{2\mathrm{Re}(h)}. \tag{3.5}$$

Letting $f := e^{-\tilde a}\bar c_w$, from the second equation in (3.4), one can see that

$$f_{\bar w} = e^{-\tilde a}\bar c_{\bar w w} - e^{-\tilde a}\tilde a_{\bar w}\bar c_w = e^{-\tilde a}(\overline{c_{\bar w w} - \tilde a_w c_{\bar w}}) = 0.$$

Namely, f is holomorphic on D and

$$c_{\bar w} = e^{2\mathrm{Re}(h)}\bar f. \tag{3.6}$$

Finally, plugging (3.6) into the last equation in (3.4) we have

$$d_{\bar w w} = e^{2\mathrm{Re}(h)}|f|^2 + e^{-2\mathrm{Re}(h)}. \tag{3.7}$$

Next we verify that the curvature tensor with respect to the Kähler metric $g = \partial\bar\partial u$ is zero for such solutions with $b_{\bar w} = 0$. Substituting (3.5)-(3.7) into (3.1), the corresponding Kähler metric becomes

$$g = \begin{bmatrix} g_{1\bar 1} & g_{1\bar 2} \\ g_{2\bar 1} & g_{2\bar 2} \end{bmatrix} := \begin{bmatrix} e^h e^{\bar h} & e^h e^{\bar h}\overline{h'}\bar z + e^h e^{\bar h}\bar f \\ e^h e^{\bar h} h' z + e^h e^{\bar h} f & e^h e^{\bar h}|h'|^2|z|^2 + e^h e^{\bar h} h'\bar f z + e^h e^{\bar h}\overline{h'} f \bar z + e^h e^{\bar h}|f|^2 + e^{-h}e^{-\bar h} \end{bmatrix}.$$

Then the inverse matrix g^{-1} of g is

$$g^{-1} = \begin{bmatrix} g^{\bar{1}1} & g^{\bar{1}2} \\ g^{\bar{2}1} & g^{\bar{2}2} \end{bmatrix} = \begin{bmatrix} e^h e^{\bar{h}} |h'|^2 |z|^2 + e^h e^{\bar{h}} h' \bar{f} z + e^h e^{\bar{h}} \overline{h'} f \bar{z} + e^h e^{\bar{h}} |f|^2 + e^{-h} e^{-\bar{h}} & -e^h e^{\bar{h}} \overline{h'} \bar{z} - e^h e^{\bar{h}} \bar{f} \\ -e^h e^{\bar{h}} h' z - e^h e^{\bar{h}} f & e^h e^{\bar{h}} \end{bmatrix}.$$

Here for any holomorphic function g of the variable w, we use the notation $g' := \frac{\partial g}{\partial w}$.

Making use of the formula (3.2) we compute all the Christoffel symbols as follows.

$$\begin{aligned}
\Gamma^1_{11} =& \frac{\partial g_{1\bar{1}}}{\partial z} g^{\bar{1}1} + \frac{\partial g_{1\bar{2}}}{\partial z} g^{\bar{2}1} = 0; \\
\Gamma^1_{12} =& \frac{\partial g_{2\bar{1}}}{\partial z} g^{\bar{1}1} + \frac{\partial g_{2\bar{2}}}{\partial z} g^{\bar{2}1} = e^h e^{\bar{h}} h' \left(e^h e^{\bar{h}} |h'|^2 |z|^2 + e^h e^{\bar{h}} h' \bar{f} z + e^h e^{\bar{h}} \overline{h'} f \bar{z} + e^h e^{\bar{h}} |f|^2 + e^{-h} e^{-\bar{h}} \right) + \\
& + e^h e^{\bar{h}} h' \left(\overline{h'} \bar{z} + \bar{f} \right) \left(-e^h e^{\bar{h}} h' z - e^h e^{\bar{h}} f \right) = h'; \\
\Gamma^1_{21} =& \frac{\partial g_{1\bar{1}}}{\partial w} g^{\bar{1}1} + \frac{\partial g_{1\bar{2}}}{\partial w} g^{\bar{2}1} = \Gamma^1_{21} = h'; \\
\Gamma^1_{22} =& \frac{\partial g_{2\bar{1}}}{\partial w} g^{\bar{1}1} + \frac{\partial g_{2\bar{2}}}{\partial w} g^{\bar{2}1} = e^h e^{\bar{h}} \left((h')^2 z + h'' z + h' f + f' \right) \left[e^h e^{\bar{h}} (|h'|^2 |z|^2 + h' \bar{f} z + \overline{h'} f \bar{z} + |f|^2) + e^{-h} e^{-\bar{h}} \right] + \\
& + \left[e^h e^{\bar{h}} \left[(h' |h'|^2 + h'' \overline{h'}) |z|^2 + ((h')^2 + h'') \bar{f} z + (|h'|^2 f + \overline{h'} f') \bar{z} + (h' |f|^2 + f' \bar{f}) \right] + e^{-h} e^{-\bar{h}} (-h') \right] \\
& \left(-e^h e^{\bar{h}} h' z - e^h e^{\bar{h}} f \right) = 2(h' f + (h')^2 z) + h'' z + f'; \\
\Gamma^2_{11} =& \frac{\partial g_{1\bar{1}}}{\partial z} g^{\bar{1}2} + \frac{\partial g_{1\bar{2}}}{\partial z} g^{\bar{2}2} = 0; \\
\Gamma^2_{12} =& \frac{\partial g_{2\bar{1}}}{\partial z} g^{\bar{1}2} + \frac{\partial g_{2\bar{2}}}{\partial z} g^{\bar{2}2} = e^h e^{\bar{h}} h' \left(-e^h e^{\bar{h}} \overline{h'} \bar{z} - e^h e^{\bar{h}} \bar{f} \right) + \left(e^h e^{\bar{h}} |h'|^2 \bar{z} + e^h e^{\bar{h}} h' \bar{f} \right) e^h e^{\bar{h}} = 0; \\
\Gamma^2_{21} =& \frac{\partial g_{1\bar{1}}}{\partial w} g^{\bar{1}2} + \frac{\partial g_{1\bar{2}}}{\partial w} g^{\bar{2}2} = \Gamma^2_{12} = 0; \\
\Gamma^2_{22} =& \frac{\partial g_{2\bar{1}}}{\partial w} g^{\bar{1}2} + \frac{\partial g_{2\bar{2}}}{\partial w} g^{\bar{2}2} = e^h e^{\bar{h}} \left((h')^2 z + h'' z + h' f + f' \right) e^h e^{\bar{h}} \left(-\overline{h'} \bar{z} - \bar{f} \right) + \\
& + \left[e^h e^{\bar{h}} \left[(h' |h'|^2 + h'' \overline{h'}) |z|^2 + ((h')^2 + h'') \bar{f} z + (|h'|^2 f + \overline{h'} f') \bar{z} + (h' |f|^2 + f' \bar{f}) \right] + e^{-h} e^{-\bar{h}} (-h') \right] e^h e^{\bar{h}} \\
=& - h'.
\end{aligned}$$

Specifically, all the Christoffel symbols are holomorphic. Since for a Kähler metric, all the curvature tensor coefficients vanish, except

$$R^\delta_{\alpha\bar{\beta}\gamma} = -\frac{\partial \Gamma^\delta_{\alpha\gamma}}{\partial \bar{z}_\beta}, R^\delta_{\bar{\alpha}\beta\gamma} = \frac{\partial \Gamma^\delta_{\beta\gamma}}{\partial \bar{z}_\alpha}, \quad \alpha, \beta, \gamma \in \{1, \ldots, n\}$$

and their conjugates, we have the curvature $R \equiv 0$. □

Theorem 3.1. *For any $R > 0$, there exist infinitely many real-analytic plurisubharmonic solutions to (1.1) on $\mathbb{C} \times D_R$ such that the corresponding Kähler metric is nowhere flat.*

Proof. Given any holomorphic function f such that $|f| < 1$ and $|f'| \neq 0$ on D_R, let

$$\tilde{a} = \ln \frac{|f'|}{1 - |f|^2} \quad \text{on} \quad D_R.$$

For example, one can choose f to be any conformal map from D_R into D_1. Apparently,

$$a = e^{\tilde{a}} = \frac{|f'|}{1-|f|^2}$$

is real-analytic on D_R.

Let b solve

$$b_{\bar{w}} = \frac{|f'|^2}{2(1-|f|^2)^2} \quad \text{on} \quad D_R.$$

Indeed, since the right hand side has a Taylor expansion at 0 with radius of convergence R, one can solve a b by term-by-term integration of the expansion at 0 using the following general formula:

$$b = \sum_{i+j=0}^{\infty} \frac{b_{ij}}{j+1} w^i \bar{w}^{j+1} \quad \text{solves} \quad b_{\bar{w}} = \sum_{i+j=0}^{\infty} b_{ij} w^i \bar{w}^j$$

on its domain of convergence. Consequently, the Taylor expansion of b has radius of convergence R. Moreover, one can directly verify that the pair $(\tilde{a}, b)$ satisfies the first two equations in (2.3). Then as in the proof of Theorem 2.3, we further let c be any holomorphic function so (2.4) is satisfied, and solve (2.5) for the Taylor expansion of d on D_R (again, by term-by-term integration). Altogether, the function u defined in (1.2) with these coefficients a, b, c and d is a real-analytic plurisubharmonic solution to (1.1) on $\mathbb{C} \times D_R$. With such chosen u, the corresponding Kähler metric is nowhere flat according to (the proof of) Theorem 1.1. □

Following the proof of Theorem 3.1, let us produce a concrete real-analytic solution to (1.1) on $\mathbb{C} \times D_R$.

Example 1. *Letting $f(w) = w/R$ on D_R, then $f' \neq 0, |f| < 1$ on D_R. Following the proof of Theorem 3.1, we have*

$$a = e^{\tilde{a}} = \frac{R}{R^2 - |w|^2}, \quad b_{\bar{w}} = \frac{R^2}{2(R^2-|w|^2)^2}.$$

In particular, in terms of Taylor expansion at $w = 0$, $b_{\bar{w}} = \sum_{n=0}^{\infty} \frac{(n+1)|w|^{2n}}{2R^{2n+2}}$. Thus the term-by-term integration gives a solution

$$b = \sum_{n=0}^{\infty} \frac{w^n \bar{w}^{n+1}}{2R^{2n+2}} = \frac{\bar{w}}{2(R^2-|w|^2)}.$$

Further letting $c = 0$ so that (2.4) is automatically satisfied. Finally we solve d so that (2.5) holds, which in this case becomes

$$d_{\bar{w}w} = e^{-\tilde{a}} = \frac{R^2 - |w|^2}{R}.$$

One such d is

$$d = R|w|^2 - \frac{|w|^4}{4R}.$$

Altogether, we obtain a solution to (1.1) on $\mathbb{C} \times D_R$:

$$u(z,w) = \frac{R|z|^2}{R^2-|w|^2} + \frac{\bar{w}z^2}{2(R^2-|w|^2)} + \frac{w\bar{z}^2}{2(R^2-|w|^2)} + R|w|^2 - \frac{|w|^4}{4R}, \tag{3.8}$$

whose induced Kähler metric is nowhere flat since $R^1_{111} = 4|b_{\bar{w}}|^2 \neq 0$ on $\mathbb{C} \times D_R$.

It is worth noting that the metric given by the complex Hessian of u in (3.8) is not complete. Indeed, the path $\gamma(t) := (0,t), 0 < t < R$ originates from the origin with the initial velocity $(0,1)$, and approaches the boundary point $(0,R)$. However, the length of γ with respect to $g := \partial\bar{\partial}u$ is

$$\int_0^R \|\dot{\gamma}\|_g dt = \int_0^R \sqrt{2g_{\bar{w}w}|_\gamma} dt = \int_0^R \sqrt{2\left(R - \frac{t^2}{R}\right)} dt = \frac{\sqrt{2}}{4}\pi R^{\frac{3}{2}} < \infty.$$

By the Hopf-Rinow Theorem, it is not complete.

In particular, Calabi's question fails for any bounded domain in $\mathbb{C}^2$, by simply restricting the solutions in Theorem 3.1 on this domain.

Corollary 3.2. *For every bounded domain $\Omega \subset \mathbb{C}^2$, there exist infinitely many real-analytic plurisubharmonic solutions to (1.1) on Ω such that its induced Kähler metric is nowhere flat.*

4 Entire solutions with flat Kähler metrics

In this section, we convert the attention to entire solutions. First we construct ample entire real-analytic plurisubharmonic solutions to (1.1) on $\mathbb{C}^2$ of the quadratic form (1.2).

Theorem 4.1. *Given any three entire holomorphic functions h, f and b in $w \in \mathbb{C}$, define*

$$a := e^{2Re(h)}$$

and let c and d be solutions to

$$c_{\bar{w}} = a\bar{f}, \quad d_{\bar{w}w} = a|f|^2 + \frac{1}{a} \quad on \quad \mathbb{C}. \tag{4.1}$$

Then the function u defined in (1.2) with the coefficients (a,b,c,d) given above is an entire real-analytic plurisubharmonic solution to (1.1) on $\mathbb{C}^2$. Moreover, the induced Kähler metric given by the solution is flat.

Proof. Since h is holomorphic on $\mathbb{C}$, the function a admits a Taylor expansion at 0 with infinite radius of convergence. Consequently, both $a\bar{f}$ and $a|f|^2 + \frac{1}{a}$ also have Taylor expansions at 0 with infinite radius of convergence. Therefore, (4.1) can be integrated term-by-term using these expansions, producing Taylor expansions of c and d with infinite radius of convergence. This establishes the existence of the coefficients c and d on $\mathbb{C}$. The fact that these particular choices yield a solution to (1.1) with a flat induced Kähler metric follows directly from the second part (when $b_{\bar{w}} \equiv 0$) in the proof to Theorem 1.1. □

In particular, this theorem generalizes the examples by Warren [17], and Myga [13, Proposition 1] where h is holomorphic, and $b = c \equiv 0$, with flat induced Kähler metrics. That is, none of our examples yields a counterexample to Calabi' s question.

Example 2. *Letting $h = w$, $f = e^{-w}$, and b be any holomorphic function (say, $b \equiv 0$) in Theorem 4.1, one can see that*

$$u(z,w) = e^{2Re(w)}|z|^2 + \bar{w}e^w z + we^{\bar{w}}\bar{z} + e^{-2Re(w)} + |w|^2$$

is an entire solution to (1.1).

Theorem 4.1 has found a special class of entire solutions to (1.1) of the form (1.2) where Calabi's open question is true. A natural question arises whether Calabi's question is true for any entire solutions of the form (1.2). Namely,

Question: Is the Kähler metric given by the complex Hessian of any smooth solution to (1.1) on $\mathbb{C}^2$ of the form (1.2) necessarily flat? By Theorem 1.1, this is further equivalent to asking: must every smooth entire solution of the form (1.2) satisfy the condition $b_{\bar{w}} \equiv 0$?

Although it remains unclear for the above question in full generality, we show that $b_{\bar{w}}$ must have zeros somewhere on $\mathbb{C}$, as a step toward addressing it. The proof makes use of a non-existence result of entire solutions by Osserman [14].

Theorem 4.2. *For every C^2-smooth entire solution to (1.1) on $\mathbb{C}^2$ of the form (1.2), the zero set $Z_b := \{w \in \mathbb{C} : b_{\bar{w}}(w) = 0\}$ of $b_{\bar{w}}$ must be nonempty.*

Proof. Assume by contradiction that there exists a C^2 solution of the form (1.2) such that $b_{\bar{w}}$ is nowhere zero on $\mathbb{C}$. Let us revisit the original system (2.3). Letting $h := e^{-2\tilde{a}} b_{\bar{w}}$ and making use of the equation $b_{\bar{w}w} = 2\tilde{a}_w b_{\bar{w}}$, one has h satisfies

$$h_w = e^{-2\tilde{a}} b_{\bar{w}w} - 2\tilde{a}_w e^{-2\tilde{a}} b_{\bar{w}} = e^{-2\tilde{a}} 2\tilde{a}_w b_{\bar{w}} - 2\tilde{a}_w e^{-2\tilde{a}} b_{\bar{w}} = 0 \quad \text{on} \quad \mathbb{C}.$$

Namely, h is anti-holomorphic on $\mathbb{C}$, with

$$b_{\bar{w}} = e^{2\tilde{a}} h \quad \text{on} \quad \mathbb{C}. \tag{4.2}$$

Moreover, since $b_{\bar{w}} \neq 0$, h is nowhere zero on $\mathbb{C}$.

Plugging it to the equation for $\tilde{a}$, we obtain

$$\tilde{a}_{\bar{w}w} = 4|h|^2 e^{2\tilde{a}} \quad \text{on} \quad \mathbb{C}. \tag{4.3}$$

We claim there is no entire solution to (4.3). If not, since h is nowhere zero on $\mathbb{C}$, $\ln(4|h|)$ is harmonic on $\mathbb{C}$, and thus $v := \ln(4|h|) + \tilde{a}$ is a C^2 solution to

$$\Delta v = e^{2v} \quad \text{on} \quad \mathbb{C}. \tag{4.4}$$

However, we recall a nonexistence result of Osserman [14] for solutions to $\Delta u = f(u)$ on $\mathbb{C}$: if $f > 0, f' \geq 0$ on $\mathbb{R}$ and

$$\int_0^\infty \left(\int_0^t f(s)ds \right)^{-\frac{1}{2}} dt < \infty,$$

then there is no C^2 solution to $\Delta u = f(u)$ on $\mathbb{C}$. One can check that $f(s) := e^{2s}$ in our case satisfies the above assumptions for f. From this we immediately obtain the nonexistence of entire solutions to (4.4). But this is a contradiction! The claim is thus proved, and so is the theorem. □

Unfortunately the approach used in the proof of Theorem 4.2 no longer works if the function h there has zeros. This is because the singularity of v cannot be resolved at the zeros of h to yield a (weak) solution to (4.4) everywhere. In the Appendix we will discuss several cases where the isolated singularities can, in fact, be resolved. On the other hand, note that due to the holomorphy of h, the zero set Z_b of $b_{\bar{w}}$, if not empty, is either isolated or the whole $\mathbb{C}$ from (4.2). Therefore to answer **Question**, by Theorem 4.2 it suffices to consider the case when Z_b is isolated.

5 Solutions that are radially symmetric on w

In this section, we investigate solutions to (1.1) that depends quadratically on the variable z and exhibit radial symmetry in the variable w. More precisely, the solutions are of the form (1.2), where a, b, c and d depend solely on $|w|$. In the following, we derive explicit formulas for all these functions. Interestingly, such solutions exhibit a wide range of singular behaviors.

Letting $t := \log |w|^2$, we shall explore all possible forms of expressions of a, b, c and d in terms of the variable t. For a radial function h of the variable w, by the chain rule, $h_w = \frac{h_t}{w}$, $h_{\bar{w}} = \frac{h_t}{\bar{w}}$, and $h_{\bar{w}w} = \frac{h_{tt}}{|w|^2}$ away from $w = 0$. Here h_t denotes the derivative of h with respect to t. In view of this, (2.1) can be reduced to a system of ordinary differential equations:

$$\begin{cases} aa_{tt} = |a_t|^2 + 4|b_t|^2; \\ ab_{tt} = 2a_t b_t; \\ ac_{tt} = a_t c_t + 2b_t \bar{c}_t; \\ ad_{tt} = |c_t|^2 + e^t. \end{cases} \tag{5.1}$$

To solve the system, let $\tilde{b} := b_t$, so the second equation in (5.1) becomes

$$a\tilde{b}_t = 2a_t\tilde{b}.$$

Separating the two functions and then taking integration on both sides, one gets

$$\tilde{b}\ (= b_t) = ka^2 \tag{5.2}$$

for any complex constant k.

Plugging (5.2) into the first equation in (5.1), we seek solutions to

$$aa_{tt} - a_t^2 = 4|k|^2 a^4. \tag{5.3}$$

Let $v := \frac{a_t}{a}$, or equivalently

$$a_t = av. \tag{5.4}$$

Then $v_t = \frac{aa_{tt} - a_t^2}{a^2}$, and (5.3) can be simplified to a first-order ODE in terms of v:

$$v_t = 4|k|^2 a^2 \tag{5.5}$$

Now that a coupled system (5.4)-(5.5) is established about (a, v), by separating the variables, we get

$$v\ dv = 4|k|^2 a\ da.$$

Integrating both sides, one can eventually obtain

$$v^2 = 4|k|^2 a^2 + C_1,$$

for any real constant C_1 (since a and v are real) such that the right hand side is nonnegative. Recalling that $v = \frac{a_t}{a}$, hence

$$a_t = \pm a\sqrt{4|k|^2 a^2 + C_1} \tag{5.6}$$

for any real constant C_1 with $4|k|^2 a^2 + C_1 \geq 0$, and for the constant k chosen in (5.2). Depending on the value of k, and then that of C_1, we explore all possible expressions of a, b, c and d below.

Case I: $k = 0$**.** A straight forward computation from (5.6) and (5.2) gives

$$a = C_1 e^{C_2 t}, \quad b = C_3 \tag{5.7}$$

for any real constants $C_1 > 0$ (since $a > 0$) and C_2, and any complex constant C_3. Plugging them into the third and fourth equations in (5.1), we have

$$\begin{aligned} &c = C_4 e^{C_2 t} + C_5, \\ &d = \begin{cases} \dfrac{|C_4|^2 e^{C_2 t}}{C_1} + \dfrac{e^{(1-C_2)t}}{(1-C_2)^2 C_1} + C_6 t + C_7, & C_2 \neq 1, \\ \dfrac{|C_4|^2 e^{C_2 t}}{C_1} + \dfrac{t^2}{2C_1} + C_6 t + C_7, & C_2 = 1, \end{cases} \end{aligned} \tag{5.8}$$

where $C_1 > 0$, C_2, C_6 and C_7 are any real constants, and C_3, C_4 and C_5 are any complex constants.

Case II: $k \neq 0$**.** Rewrite (5.6) to be

$$a_t = \pm 2|k| a \sqrt{a^2 + C_1}, \tag{5.9}$$

where C_1 is any real constant such that $a^2 + C \geq 0$. Separate the variables in (5.9) and then integrate both sides to get

$$\pm \int \frac{da}{a\sqrt{a^2 + C_1}} = 2|k|t + C \tag{5.10}$$

for any real constant C. Depending on the sign of C_1, there are three cases for the integral on the left hand side of (5.10):

$$\int \frac{da}{a\sqrt{a^2+C_1}} = \begin{cases} \dfrac{1}{\sqrt{C_1}} \ln \left(\dfrac{\sqrt{a^2 + C_1} - \sqrt{C_1}}{a} \right) + C, & C_1 > 0; \\ -\dfrac{1}{a} + C, & C_1 = 0; \\ \dfrac{1}{\sqrt{-C_1}} \sec^{-1} \left(\dfrac{a}{\sqrt{-C_1}} \right) + C, & C_1 < 0. \end{cases}$$

Combining these formulas with (5.10), we can solve a explicitly:

$$a \text{ (or } -a) = \begin{cases} \dfrac{2\sqrt{C_1} C_2 e^{2|k|\sqrt{C_1}t}}{1 - C_2^2 e^{4|k|\sqrt{C_1}t}}, & C_1 > 0; \\ -\dfrac{1}{2|k|t + C_2}, & C_1 = 0; \\ \sqrt{-C_1} \sec\left(2|k|\sqrt{-C_1}t + C_2\right), & C_1 < 0, \end{cases} \tag{5.11}$$

where C_2 is an arbitrary real constant. The domain of definition for a is wherever the expression on the right hand side is defined such that $a > 0$.

Next we solve for b. Recalling that $b_t = ka^2$ and combining it with the expression (5.11) of a, one can immediately obtain

$$b = \begin{cases} \dfrac{\sqrt{C_1} k}{|k|(1 - C_2^2 e^{4|k|\sqrt{C_1}t})} + k_0, & C_1 > 0; \\ -\dfrac{k}{2|k|(2|k|t + C_2)} + k_0, & C_1 = 0; \\ \dfrac{k\sqrt{-C_1}}{2|k|} \tan\left(2|k|\sqrt{-C_1}t + C_2\right) + k_0, & C_1 < 0, \end{cases} \tag{5.12}$$

where C_2 is the real constant chosen in (5.11), and k_0 is an arbitrary complex constant.

Plugging expressions of a and b (in particular, (5.2) and (5.9)) into the third equation in (5.1) for c, we further solve

$$c_{tt} = \pm 2|k|\sqrt{a^2 + C_1}c_t + 2ka\bar{c}_t.$$

Writing $c = c_1 + ic_2$, where c_1, c_2 are real functions, and $k = k_1 + ik_2$, where k_1, k_2 are real constants, and separating the real parts from the imaginary parts in the above expression, we get

$$\begin{cases} (c_1)_{tt} = \pm 2|k|\sqrt{a^2 + C_1}(c_1)_t + 2a\left(k_1(c_1)_t + k_2(c_2)_t\right); \\ (c_2)_{tt} = \pm 2|k|\sqrt{a^2 + C_1}(c_2)_t + 2a\left(k_2(c_1)_t - k_1(c_2)_t\right). \end{cases} \tag{5.13}$$

This can be rewritten in a matrix form as follows:

$$\begin{pmatrix} (c_1)_{tt} \\ (c_2)_{tt} \end{pmatrix} = A \begin{pmatrix} (c_1)_t \\ (c_2)_t \end{pmatrix},$$

where

$$A = \pm 2|k|\sqrt{a^2 + C_1}I_2 + 2aK, \text{ with } \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_1 & k_2 \\ k_2 & -k_1 \end{pmatrix}.$$

Note that the constant matrix K has eigenvalues $\lambda = \pm\sqrt{k_1^2 + k_2^2} = \pm|k|$. So there exists a constant unitary matrix P that diagonalizes K:

$$K = P \begin{pmatrix} |k| & 0 \\ 0 & -|k| \end{pmatrix} P^{-1}.$$

Defining the new variables $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = P^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, the system can be decoupled into:

$$\begin{cases} (w_1)_{tt} = 2|k|\left(\pm\sqrt{a^2 + C_1} + a\right)(w_1)_t; \\ (w_2)_{tt} = 2|k|\left(\pm\sqrt{a^2 + C_1} - a\right)(w_2)_t. \end{cases}$$

Thus, without loss of generality, let us assume for simplicity that k is a positive constant and the "+" sign in (5.13) is taken. It then can be rewritten as

$$\begin{cases} (\ln|(c_1)_t|)_t = 2k(\sqrt{a^2 + C_1} + a); \\ (\ln|(c_2)_t|)_t = 2k(\sqrt{a^2 + C_1} - a). \end{cases} \tag{5.14}$$

Making use of (5.11), which further gives

$$\sqrt{a^2 + C_1} = \begin{cases} \dfrac{\sqrt{C_1}(1 + C_2^2 e^{4|k|\sqrt{C_1}t})}{1 - C_2^2 e^{4|k|\sqrt{C_1}t}}, & C_1 > 0; \\ -\dfrac{1}{2|k|t + C_2}, & C_1 = 0; \\ \sqrt{-C_1}\tan\left(2\sqrt{-C_1}|k|t + C_2\right), & C_1 < 0, \end{cases}$$

we take integration directly on both sides of (5.14) to have

$$(c_1)_t = \begin{cases} \dfrac{C_{3,1}e^{2k\sqrt{C_1}t}}{\left(1 - C_2 e^{2k\sqrt{C_1}t}\right)^2}; \\ \dfrac{C_{3,1}}{(2kt + C_2)^2}; \\ \dfrac{C_{3,1}}{1 - \sin\left(2\sqrt{-C_1}|k|t + C_2\right)}, \end{cases} \qquad (c_2)_t = \begin{cases} \dfrac{C_{3,2}e^{2k\sqrt{C_1}t}}{\left(1 + C_2 e^{2k\sqrt{C_1}t}\right)^2}, & C_1 > 0; \\ C_{3,2}, & C_1 = 0; \\ \dfrac{C_{3,2}}{1 + \sin\left(2\sqrt{-C_1}|k|t + C_2\right)}, & C_1 < 0, \end{cases} \tag{5.15}$$

where $C_{3,1}, C_{3,2}$ are any real constants.

Integrating one more time in (5.15), we get

$$c_1 = \begin{cases} \dfrac{C_{3,1}}{2k\sqrt{C_1}C_2(1 - C_2 e^{2k\sqrt{C_1}t})} + C_{4,1}, & C_1 > 0; \\ -\dfrac{C_{3,1}}{2k(2kt + C_2)} + C_{4,1}, & C_1 = 0; \\ \dfrac{C_{3,1}}{2\sqrt{-C_1}k} \left(\sec\left(2\sqrt{-C_1}kt + C_2\right) + \tan\left(2\sqrt{-C_1}kt + C_2\right)\right) + C_{4,1}, & C_1 < 0, \end{cases}$$

and

$$c_2 = \begin{cases} -\dfrac{C_{3,2}}{2k\sqrt{C_1}C_2(1 + C_2 e^{2k\sqrt{C_1}t})} + C_{4,2}, & C_1 > 0; \\ C_{3,2}t + C_{4,2}, & C_1 = 0; \\ -\dfrac{C_{3,2}}{2\sqrt{-C_1}k} \left(\sec\left(2\sqrt{-C_1}kt + C_2\right) - \tan\left(2\sqrt{-C_1}kt + C_2\right)\right) + C_{4,2}, & C_1 < 0, \end{cases}$$

where $C_{4,1}, C_{4,2}$ are any real constants. Thus

$$c = c_1 + ic_2 = \begin{cases} \dfrac{\overline{C_3} + C_3 C_2 e^{2k\sqrt{C_1}t}}{2k\sqrt{C_1}C_2(1 - C_2^2 e^{4k\sqrt{C_1}t})} + C_4, & C_1 > 0; \\ -\dfrac{\mathrm{Re}(C_3)}{2k(2kt + C_2)} + i\mathrm{Im}(C_3)t + C_4, & C_1 = 0; \\ \dfrac{\overline{C_3}}{2\sqrt{-C_1}k} \sec\left(2\sqrt{-C_1}kt + C_2\right) + \dfrac{C_3}{2\sqrt{-C_1}k} \tan\left(2\sqrt{-C_1}kt + C_2\right) + C_4, & C_1 < 0, \end{cases} \tag{5.16}$$

where $k > 0$ and C_2 are defined in (5.11) and (5.12), and $C_3 := C_{3,1} + iC_{3,2}$ and $C_4 := C_{4,1} + iC_{4,2}$ are arbitrary complex constants.

Finally, one can substitute the expression (5.11) of a, and (5.15) of c_t into the fourth equation in (5.1) to solve for d:

$$d = \begin{cases} \dfrac{e^{(1-2k\sqrt{C_1})t}}{2\sqrt{C_1}C_2(1 - 2k\sqrt{C_1})^2} - \dfrac{C_2 e^{(1+2k\sqrt{C_1})t}}{2\sqrt{C_1}(1 + 2k\sqrt{C_1})^2} + \\ + \dfrac{1}{8k^2 C_1^{\frac{3}{2}} C_2} \left(\dfrac{C_{3,1}^2}{1 - C_2 e^{2k\sqrt{C_1}t}} - \dfrac{C_{3,2}^2}{1 + C_2 e^{2k\sqrt{C_1}t}}\right) + C_5 t + C_6, & C_1 > 0; \\ -(2kt + C_2 - 4k)e^t - \dfrac{C_{3,1}^2}{8k^2(2kt + C_2)} - C_{3,2}^2 \left(\dfrac{kt^3}{3} + \dfrac{C_2 t^2}{2}\right) + C_5 t + C_6, & C_1 = 0; \\ \dfrac{e^t \left[(1 + 4k^2 C_1)\cos\left(2k\sqrt{-C_1}t + C_2\right) + 4k\sqrt{-C_1}\sin\left(2k\sqrt{-C_1}t + C_2\right)\right]}{\sqrt{-C_1}(1 - 4k^2 C_1)^2} + \\ + \dfrac{(C_{3,1}^2 - C_{3,2}^2)\tan\left(2k\sqrt{-C_1}t + C_2\right) + (C_{3,1}^2 + C_{3,2}^2)\sec\left(2k\sqrt{-C_1}t + C_2\right)}{4k^2 C_1} + C_5 t + C_6, & C_1 = 0, \end{cases} \tag{5.17}$$

for $k > 0$, C_2 and $C_{3,1}, C_{3,2}, C_{4,1}, C_{4,2}$ defined in (5.11), (5.12) and (5.16) with $C_3 = C_{3,1} + iC_{3,2}$ and $C_4 = C_{4,1} + iC_{4,2}$, and any real constants C_5 and C_6.

Based on the above derivation for solutions to (1.1) that depend radially on w, choosing different values for the parameters yields the following interesting examples with varying singularities. Note that the expressions for c do not introduce any additional singularities beyond those arising from

a, b or $t = \ln|w|^2$. For the purpose of studying singularity of solutions, we will consider the special cases when $C_3 = C_4 = 0$, which significantly simplifies the expression for d. We also let $k_0 = C_5 = C_6 = 0$ below.

Example 3. *Taking $k = 0$ and $C_1 = 1$, $C_2 \neq 1$ in (5.7) and (5.8), we obtain a solution to (1.1):*

$$u = |w|^{2C_2}|z|^2 + (z^2 + \bar{z}^2) + \frac{|w|^{2-2C_2}}{(1-C_2)^2}.$$

Different choices of C_2 lead to solutions with different regularity. For instance, if $C_2 = 0$, then u is a (smooth) quadratic function

$$u = |z|^2 + (z^2 + \bar{z}^2) + |w|^2;$$

if $C_2 = \frac{3}{2}$, then u has singularity at $w = 0$:

$$u = |w|^3|z|^2 + (z^2 + \bar{z}^2) + \frac{4}{|w|};$$

if $C_2 = \frac{1}{2}$, then u is Lipschitz at $w = 0$:

$$u = |w||z|^2 + (z^2 + \bar{z}^2) + 4|w|.$$

In particular, the last function coincides with an example of Błocki [2] and He [8] when $n = 2$, who showed it is both a pluripotential and viscosity solution.

Example 4. *Taking $k > 0$ and $C_1 = C_2 = 1$ in (5.11), (5.12) and (5.17), we obtain a solution to (1.1):*

$$u = \frac{2|w|^{4k}}{1-|w|^{8k}}|z|^2 + \frac{1}{1-|w|^{8|k|}}(z^2 + \bar{z}^2) + \frac{|w|^{2-4k}}{2(1-2k)^2} - \frac{|w|^{2+4k}}{2(1+2k)^2}.$$

Different choices of k lead to solutions with different regularity. For instance, if $k = 1$, then u blows up at $w = 0$ and $|w| = 1$:

$$u = \frac{2|w|^4}{1-|w|^8}|z|^2 + \frac{1}{1-|w|^8}(z^2 + \bar{z}^2) + \frac{1}{2|w|^2} - \frac{|w|^6}{18};$$

if $k = \frac{1}{4}$, then u is Lipschitz at $w = 0$, and blows up at $|w| = 1$:

$$u = \frac{2|w|}{1-|w|^2}|z|^2 + \frac{1}{1-|w|^2}(z^2 + \bar{z}^2) + 2|w| - \frac{2|w|^3}{9},$$

which is a pluripotential and viscosity solution on $\mathbb{C} \times D_1$, similar as in [2] and [8].

Example 5. *Taking $k = 1$ and $C_1 = C_2 = 0$ in (5.11), (5.12) and (5.17), we obtain a solution to (1.1):*

$$u = -\frac{1}{2\ln(|w|^2)}|z|^2 - \frac{1}{4\ln(|w|^2)}(z^2 + \bar{z}^2) - \left(2\ln(|w|^2) - 4\right)|w|^2 \quad \text{on} \quad \mathbb{C} \times D_1.$$

This is the example of a pluripotential and viscosity solution given by Wang-Wang [18]. This solution is in $W^{1,2}_{loc} \cap W^{2,1}_{loc}$ but fails to be in $W^{1,p}_{loc}$ for any $p > 2$, or in $W^{2,q}_{loc}$ for any $q > 1$. Moreover, it is not even Dini continuous near $w = 0$.

Example 6. *Taking $k = -C_1 = 1$ and $C_2 = 0$ in (5.11), (5.12) and (5.17), we obtain a solution to (1.1):*

$$u = \frac{1}{\cos(2\ln(|w|^2))}|z|^2 + \frac{\tan(2\ln(|w|^2))}{2}(z^2 + \bar{z}^2) + \frac{|w|^2\left(-3\cos(2\ln(|w|^2)) + 4\sin(2\ln(|w|^2))\right)}{25}.$$

This solution exhibits singularities at $|w| = e^{\frac{\pi}{8}}e^{\frac{k\pi}{4}}$ for each $k \in \mathbb{Z}$. In particular, when $k \to -\infty$, the singularity set accumulates to $w = 0$. Moreover, this solution is no longer plurisubharmonic due to the frequent sign change.

We conclude the section with an example of a solution to (1.1) whose singularity set is of real codimension-one.

Example 7. *It is straightforward to see that*

$$u(z, w) = \begin{cases} |z-1|^2 + |w|^2, & Re(z) \le \frac{1}{2}; \\ |z|^2 + |w|^2, & Re(z) > \frac{1}{2} \end{cases} \tag{5.18}$$

solves (1.1) on $\mathbb{C}^2 \setminus X$, where $X := \{(z, w) \in \mathbb{C}^2 : Re(z) = \frac{1}{2}\}$ is a real hypersurface of real codimension-one. Note that since u is is plurisubharmonic and continuous everywhere, the fundamental result [3] of Bedford and Taylor guarantees that $(dd^c u)^2$ is well-defined as a positive Borel measure on $\mathbb{C}^2$, where $d = \bar{\partial} + \partial$, $d^c = \frac{i}{2}(\bar{\partial} - \partial)$. In detail, letting dV be the volume form on $\mathbb{C}^2$, and dS be the surface measure on X, then the measure is

$$(dd^c u)^2 = 4dV + 2\lambda_X \quad on \quad \mathbb{C}^2,$$

where λ_X is a Lelong current over X defined by $\lambda_X(\phi) = \int_X \phi dS$ for any testing function ϕ. In particular, the example shows that the singularity of u at X is not removable for (1.1).

6 Rigidity of more general solutions

In the previous sections, we considered solutions dependent quadratically on the variable z. A natural question is to consider a more general form of solutions by, say, replacing z in (1.2) by a holomorphic function ϕ of z. That is,

$$u(z, w) = a(w)|\phi(z)|^2 + b(w)\phi^2(z) + \overline{b(w)\phi^2(z)} + c(w)\phi(z) + \overline{c(w)\phi(z)} + d(w). \tag{6.1}$$

The complex Hessian of u in (6.1) becomes

$$\partial\bar{\partial}u = \begin{bmatrix} u_{\bar{z}z} & u_{z\bar{w}} \\ u_{\bar{z}w} & u_{\bar{w}w} \end{bmatrix} = \begin{bmatrix} a|\phi'|^2 & a_{\bar{w}}\phi'\bar{\phi} + 2b_{\bar{w}}\phi'\phi + c_{\bar{w}}\phi' \\ a_w\overline{\phi'}\phi + 2\bar{b}_w\overline{\phi\phi'} + \bar{c}_w\overline{\phi'} & a_{\bar{w}w}|\phi|^2 + b_{\bar{w}w}\phi^2 + \overline{b_{\bar{w}w}\phi^2} + c_{\bar{w}w}\phi + \overline{c_{\bar{w}w}\phi} + d_{\bar{w}w} \end{bmatrix}. \tag{6.2}$$

Due to the strict plurisubharmonicity of u, one has $\phi' \neq 0$ necessarily. By further adjusting the coefficients (a, b, c, d), we can assume that ϕ is normalized such that

$$\phi(0) = 0 \quad \text{and} \quad \phi'(0) = 1.$$

In order for (6.1) to solve (1.1), the following theorem demonstrates that the choice of ϕ takes very rigid forms.

Theorem 6.1. *Let u be a smooth solution to (1.1) of the form (6.1) for some holomorphic function ϕ of z such that $\phi(0) = 0$, $\phi' \neq 0$ and $\phi'(0) = 1$, then either*

$$\phi = z \quad on \quad \mathbb{C},$$

or there exists some nonzero constant α such that

$$\phi = \frac{1 - \sqrt{1 - 2\alpha z}}{\alpha} \quad on \quad D_{\frac{1}{2|\alpha|}}, \tag{6.3}$$

where the complex square root takes the principal branch.

Proof. Taking the determinant of the Hessian (6.2) and applying (1.1), we obtain

$$\begin{aligned} |\phi'|^{-2}(= |\phi'|^{-2}\det(\partial\bar{\partial}u)) =& |\phi|^2 \left(a a_{\bar{w}w} - |a_w|^2 - 4|b_{\bar{w}}|^2\right) + \phi^2 \left(a b_{\bar{w}w} - 2a_w b_{\bar{w}}\right) + \overline{\phi^2} \left(a \overline{b_{\bar{w}w}} - 2a_{\bar{w}}\bar{b}_w\right) \\ &+ \phi \left(a c_{\bar{w}w} - a_w c_{\bar{w}} - 2b_{\bar{w}}\bar{c}_w\right) + \bar{\phi}\left(a\bar{c}_{\bar{w}w} - a_{\bar{w}}\bar{c}_w - 2\bar{b}_w c_{\bar{w}}\right) + \left(a d_{\bar{w}w} - |c_{\bar{w}}|^2\right). \end{aligned} \tag{6.4}$$

Write both sides in terms of the Taylor expansion of z at 0. We first collect terms without z on both sides by letting $z = 0$. Recalling the normalization assumptions on ϕ, this gives

$$a d_{\bar{w}w} - |c_{\bar{w}}|^2 = 1. \tag{6.5}$$

Collecting the coefficients of the term z in (6.4), one has

$$a c_{\bar{w}w} - a_w c_{\bar{w}} - 2b_{\bar{w}}\bar{c}_w = -\phi''(0). \tag{6.6}$$

Collecting the coefficients of the term z^2 in (6.4), one has

$$a b_{\bar{w}w} - 2a_w b_{\bar{w}} + \frac{\phi''(0)}{2}(a c_{\bar{w}w} - a_w c_{\bar{w}} - 2b_{\bar{w}}\bar{c}_w) = -\frac{\phi'''(0)}{2} + (\phi''(0))^2.$$

Combined with (6.6), one futher gets

$$a b_{\bar{w}w} - 2a_w b_{\bar{w}} = \frac{3(\phi''(0))^2 - \phi'''(0)}{2}. \tag{6.7}$$

Collecting the coefficients of the term $|z|^2$ in (6.4), one has

$$a a_{\bar{w}w} - |a_w|^2 - 4|b_{\bar{w}}|^2 = |\phi''(0)|^2. \tag{6.8}$$

Substituting (6.5), (6.6), (6.7) and (6.8) into (6.4), we obtain

$$|\phi'|^{-2} = |\phi''(0)|^2|\phi|^2 + \frac{3(\phi''(0))^2 - \phi'''(0)}{2}\phi^2 + \overline{\frac{3(\phi''(0))^2 - \phi'''(0)}{2}\phi^2} - \phi''(0)\phi - \overline{\phi''(0)\phi} + 1. \tag{6.9}$$

Denoting $g := (\phi')^{-1}$ and taking Δ on both sides of (6.9), one gets $|g'|^2 = |\phi''(0)|^2|\phi'|^2$, or equivalently,

$$\left|\frac{g'}{\phi'}\right| \equiv |\phi''(0)|.$$

Applying the Maximum Principle to the holomorphic function g'/ϕ', we infer $g' = e^{i\theta}|\phi''(0)|\phi'$ for some $\theta \in [0, 2\pi)$. Plugging $g = (\phi')^{-1}$ in, one further sees that

$$-\frac{\phi''}{(\phi')^2} = e^{i\theta}|\phi''(0)|\phi'.$$

Evaluating the above at 0, we get $e^{i\theta}|\phi''(0)| = -\phi''(0)$. Thus $\phi'' = \phi''(0)(\phi')^3$. This is equivalent to $((\phi')^{-2})' = -2\phi''(0)$, and so

$$\phi' = \frac{1}{\sqrt{1 - 2\phi''(0)z}}.$$

Consequently, either $\phi''(0) = 0$ in which case $\phi = z$, or (6.3) holds with α there equal to $\phi''(0)$. In the latter case, one further computes that $\phi'''(0) = 3(\phi''(0))^2$. Plugging this and (6.3) back to (6.9), after simplification we immediately observe that the equation is satisfied everywhere on D_R with $R = \frac{1}{2|\alpha|}$. The proof is complete. □

As in the case for $\phi = z$, the following theorem shows there are many nontrivial solutions with ϕ taking the form (6.3).

Theorem 6.2. *Let Ω be a bounded domain in $\mathbb{C}^2$. There exists infinitely many smooth solutions to (1.1) of the form (6.1) with ϕ defined in (6.3) on Ω.*

Proof. By the rescaling method mentioned in Section 2, it suffices to construct a solution of the form (6.1) in a small neighborhood of 0.

In the case when (6.3) is taken, for instance with $\alpha = 1$, in order to obtain a solution to (1.1) of the form (6.1) near 0, according to (6.5), (6.6), (6.7) and (6.8), one looks for the coefficients a, b, c and d to satisfy

$$\begin{cases} aa_{\bar{w}w} = |a_w|^2 + 4|b_{\bar{w}}|^2 + 1; \\ ab_{\bar{w}w} = 2a_w b_{\bar{w}}; \\ ac_{\bar{w}w} = a_w c_{\bar{w}} + 2b_{\bar{w}}\bar{c}_w - 1; \\ ad_{\bar{w}w} = |c_{\bar{w}}|^2 + 1. \end{cases}$$

As before letting $\tilde{a} = \ln a$, then

$$\begin{cases} \tilde{a}_{\bar{w}w} = 4e^{-2\tilde{a}}|b_{\bar{w}}|^2 + e^{-2\tilde{a}}; \\ b_{\bar{w}w} = 2\tilde{a}_w b_{\bar{w}}; \\ c_{\bar{w}w} = \tilde{a}_w c_{\bar{w}} + 2e^{-\tilde{a}} b_{\bar{w}}\bar{c}_w - e^{-\tilde{a}}; \\ d_{\bar{w}w} = e^{-\tilde{a}}\left(|c_{\bar{w}}|^2 + 1\right). \end{cases} \tag{6.10}$$

Making use of Theorem 2.1, one can obtain infinitely many solutions to (6.10) near a neighborhood of 0. These solutions yield infinitely many solutions to (1.1) of the form (6.1) in a small neighborhood of 0. □

7 Donaldson's equation

In this section, we investigate solutions to Donaldson's equation (1.3). Donaldson's operator is strictly elliptic when $u_{tt} > 0, \Delta u > 0$ and $u_{tt}\Delta u - |\nabla u_t|^2 > 0$. He constructed in [7] infinitely many entire solutions to (1.3) on $\mathbb{R} \times \mathbb{R}^m$ of the form

$$a_0 t^2 + b(x)t + c(x), \tag{7.1}$$

where a_0 is a positive constant, and b and c are smooth real functions on $x \in \mathbb{R}^m$. He showed that every solution of the form (7.1) must satisfy

$$\Delta b = 0, \quad \Delta c = \frac{1}{2a_0}(|\nabla b|^2 + 1).$$

Note that, according to a result of Warren [17], every convex entire solution to (1.3) must be quadratic.

We shall generalize He's idea by allowing a_0 in (7.1) to depend on $x \in \mathbb{R}^m$. That is, we consider smooth solutions of the form

$$u(t,x) = a(x)t^2 + b(x)t + c(x), \tag{7.2}$$

where a, b and c are smooth real functions on $x \in \mathbb{R}^m$ with $a > 0$. Plugging it to (1.3), one gets

$$1 = u_{tt}\Delta u - |\nabla u_t|^2 = 2a(t^2\Delta a + t\Delta b + \Delta c) - |2t\nabla a + \nabla b|^2.$$

Identifying coefficients of $1, t$ and t^2, we obtain the following nonlinear differential system

$$\begin{cases} a\Delta a = 2|\nabla a|^2; \\ a\Delta b = 2\nabla a \cdot \nabla b; \\ 2a\Delta c = |\nabla b|^2 + 1. \end{cases}$$

Since $a > 0$, making use of the transformation

$$\tilde{a} = \frac{1}{a},$$

one can immediately verifies that $\tilde{a}$ satisfies

$$\Delta \tilde{a} = 0. \tag{7.3}$$

There are infinitely many positive harmonic solutions $\tilde{a}$ on B_R. For each such $\tilde{a}$, plugging $a = \frac{1}{\tilde{a}}$ into the linear elliptic equation

$$\Delta b = \frac{2}{a}\nabla a \cdot \nabla b \quad \text{on} \quad B_R \tag{7.4}$$

to solve for a smooth solution b on B_R, and then to solve

$$\Delta c = \frac{1}{2a}(|\nabla b|^2 + 1) \quad \text{on} \quad B_R \tag{7.5}$$

for a smooth c on B_R.

Theorem 7.1. *For each $R > 0$, and a harmonic function $\tilde{a}$ on B_R, define $a := \tilde{a}^{-1}$ and let b and c be solutions to (7.4)-(7.5). Then the function u defined in (7.2) with these coefficients a, b and c is a smooth solution to (1.3) on $\mathbb{R} \times B_R$.*

In the case of entire solutions, Liouville's theorem implies that every positive harmonic function on $\mathbb{R}^m$ must be a constant. Hence from (7.3) one has $\tilde{a} \equiv$ const, and further $a \equiv$ const. Moreover, (7.4)-(7.5) gives

Theorem 7.2. *Every entire solution to (1.3) on $\mathbb{R} \times \mathbb{R}^m$ of the form (7.2) must satisfy*

$$a \equiv const, \quad \Delta b = 0, \quad \Delta c = \frac{1}{2a}(|\nabla b|^2 + 1) \quad on \quad \mathbb{R}^m.$$

In particular, Theorem 7.2 states that in the case of entire solutions, (7.2) reduces to the situation (7.1) in [7]. On the other hand, in the case when $m = 1$, since the only harmonic functions are linear functions, b must be linear, and thus c must be quadratic in Theorem 7.2. One immediately obtains the following result.

Corollary 7.3. *If $m = 1$, then every entire solution to (1.3) of the form (7.2) must be a quadratic function in $(t,x) \in \mathbb{R} \times \mathbb{R}$.*

It is worth noting that Corollary 3.8 is, in fact, a special case of [17]. This is because when $m = 1$, all solutions obtained in Theorem 7.1 and Theorem 7.2 are automatically convex, due to the conditions $u_{tt} > 0$ and $\det(D^2u) = 1 > 0$.

A Resolution of isolated singularity

Let Ω be a domain in $\mathbb{R}^m, m \geq 2$. A function u is said to be a weak solution to a nonlinear differential equation

$$\Delta u = f(\cdot, u) \text{ on } \Omega,$$

if $u \in L^1_{loc}(\Omega), f(\cdot, u) \in L^1_{loc}(\Omega)$ and for any testing function $\phi \in C_c^\infty(\Omega)$, one has

$$\int_\Omega u\Delta\phi = \int_\Omega f(\cdot, u)\phi.$$

In this Appendix, we shall prove a removable singularity theorem as follows.

Theorem A.1. *Let Ω be a domain in $\mathbb{R}^m$ containing the origin, $m \geq 2$. Let $u \in C(\Omega)$ if $m = 2$, or $u \in L_{loc}^{\frac{m}{m-2}}(\Omega)$ if $m \geq 3$, and u be a weak solution to*

$$\Delta u = f(\cdot, u) \quad on \quad \Omega \setminus \{0\},$$

where $f \in C^\infty(\mathbb{R}^m \times \mathbb{R})$ with $f \geq 0$ and $\frac{\partial f}{\partial u} \geq 0$. Then $u \in C^\infty(\Omega)$ and solves

$$\Delta u = f(\cdot, u) \quad on \quad \Omega.$$

To prove the theorem, the following Harvey-Polking lemmas (see [6]) are needed for resolving the isolated singularities. Recall that D_r is the disc in $\mathbb{R}^2$ of radius r.

Lemma A.2. *If $f \in L^1(D_1)$, and $u \in C(D_1)$ is a weak solution to $\Delta u = f$ on $D_1 \setminus \{0\}$, then u is a weak solution to $\Delta u = f$ on D_1.*

Proof. Since $u - u(0) \in C(D_1)$, and is also a weak solution to $\Delta u = f$ on $D_1 \setminus \{0\}$, without loss of generality assume $u(0) = 0$. Given $0 < r < 1$, let ϕ^r be a smooth function on D_1 such that $\phi^r = 1$ on $D_{\frac{r}{2}}$, $\phi^r = 0$ outside D_r and $|\Delta\phi^r| \lesssim r^{-2}$ on D_r. Then for any testing function ϕ on D_1, $(1 - \phi^r)\phi$ is a testing function on $D_1 \setminus \{0\}$. Thus

$$\langle \Delta u - f, (1 - \phi^r)\phi \rangle = 0,$$

and so

$$\langle \Delta u - f, \phi \rangle = \langle \Delta u - f, \phi^r\phi \rangle = \langle u, \Delta(\phi^r\phi) \rangle - \langle f, \phi^r\phi \rangle.$$

Passing r to 0, since $f \in L^1(D_1)$,

$$\langle f, \phi^r\phi \rangle \lesssim \int_{D_r} |f| \to 0.$$

On the other hand,

$$\langle u, \Delta(\phi^r\phi) \rangle \lesssim r^{-2} \int_{D_r} |u| \leq \max_{D_r} |u| \to 0.$$

We thus have the desired identity $\langle \Delta u - f, \phi \rangle = 0$. □

The continuity assumption on u in Lemma A.2 can not be dropped, as demonstrated by the following example.

Example 8. *Let u be a smooth solution to*

$$\Delta u = 4e^{2u} \quad on \quad D_1.$$

For instance, one can check that $u = -\ln(1-|x|^2)$ is such a solution. Consequently, $v := u - \ln|x|$ is a smooth solution to

$$\Delta v = 4|x|^2 e^{2v} \quad on \quad D_1 \setminus \{0\}.$$

However, v is not a weak solution to

$$\Delta v = 4|x|^2 e^{2v} \quad on \quad D_1,$$

since $-\frac{1}{2\pi}\ln|x|$ is the fundamental solution to Δ. Note that $v \notin C(D_1)$, so Lemma A.2 does not apply.

By slightly adjusting the proof of Lemma A.2, one can resolve isolated singularities in higher dimensional case, under a weaker assumption on the regularity of u than continuity.

Lemma A.3. *Let B_1 be the unit ball in $\mathbb{R}^m, m \geq 3$. If $f \in L^1_{loc}(B_1)$, and $u \in L^{\frac{m}{m-2}}_{loc}(B_1)$ is a weak solution to $\Delta u = f$ on $B_1 \setminus \{0\}$, then u is a weak solution to $\Delta u = f$ on B_1.*

Proof. In view of the proof to Lemma A.2, we only need to verify that $r^{-2}\int_{B_r}|u| \to 0$. This is obvious by Hölder's inequality:

$$r^{-2}\int_{B_r}|u| \leq r^{-2}\left(\int_{B_r}|u|^{\frac{m}{m-2}}\right)^{\frac{m-2}{m}}\left(\int_{B_r}1\right)^{\frac{2}{m}} \lesssim \left(\int_{B_r}|u|^{\frac{m}{m-2}}\right)^{\frac{m-2}{m}} \to 0$$

as $r \to 0$. □

Next, we prove that under suitable smoothness and growth assumptions on f, any weak solution in fact belongs to a higher regularity class, thereby becoming a classical solution. This together with Lemma A.2 and Lemma A.3 proves Theorem A.1.

Proposition A.4. *Let Ω be a domain in $\mathbb{R}^m$. Let u be a weak solution to*

$$\Delta u = f(\cdot, u) \quad on \quad \Omega,$$

where $f \in C^\infty(\mathbb{R}^m \times \mathbb{R})$ with $f \geq 0, \frac{\partial f}{\partial u} \geq 0$. Then u is smooth on Ω.

Proof. Since $\Delta u \geq 0$ in the sense of distributions, u become subharmonic on Ω after redefining its values on a measure zero set. See, for instance, [9, Theorem 3.2.11]. Hence u is upper semi-continuous on Ω. In particular, for every $V \subset\subset \Omega$, there exists a constant $c > 0$ such that $u \leq c$ on V. Consequently, by the monotonicity of f with respect to u,

$$0 \leq f(\cdot, u) \leq f(\cdot, c) \text{ on } V.$$

This implies $\Delta u \in L^\infty(V)$.

From the $W^{2,p}$ theory of elliptic equations, we deduce that $u \in W^{2,p}(V)$ for any $p < \infty$. The Sobolev embedding theorem then yields $u \in C^{1,\alpha}(V)$ for all $0 < \alpha < 1$. Applying Schauder theory, we can further obtain $u \in C^{3,\alpha}(V)$. A standard bootstrapping argument eventually gives $u \in C^\infty(V)$. □

An immediate consequence of (the proof to) Theorem A.1 is an improved regularity for every weak solution to $\Delta u = e^{cu}$, where c is a positive constant.

Corollary A.5. *Let Ω be a domain in $\mathbb{R}^m$ and c be a positive constant. Then every weak solution to*

$$\Delta u = e^{cu} \quad on \quad \Omega$$

must be smooth on Ω.

It is natural to ask whether a similar regularity-improving property as in Theorem A.1 still holds if the condition $\frac{\partial f}{\partial u} \geq 0$ is dropped, as our approach does not extend to this case. For instance, the equation $\Delta u = e^{-u}$, where $f(u) := e^{-u}$ satisfies $\frac{\partial f}{\partial u} < 0$. The following example demonstrates that the property fails if $m \geq 3$. The situation for $m = 2$ remains unclear.

Example 9. *Let $m \geq 3$. A direct computation can verify that $u(x) = 2\ln|x| - \ln(2m-4)$ is a smooth solution to*

$$\Delta u = e^{-u} \quad on \quad \mathbb{R}^m \setminus \{0\}.$$

On the other hand, $e^{-u} = \frac{2m-4}{|x|^2} \in L^1_{loc}(\mathbb{R}^m)$, and $u \in L^p_{loc}(\mathbb{R}^m)$ for all $p < \infty$. According to Lemma A.3, u is a weak solution to

$$\Delta u = e^{-u} \quad on \quad \mathbb{R}^m.$$

However, u is not even continuous at 0.

References

[1] W. Ballmann, *Lectures on Kähler manifolds.* ESI Lect. Math. Phys. European Mathematical Society (EMS), Zürich, 2006, x+172 pp. 5

[2] Z. Błocki, *On the regularity of the complex Monge–Ampère operator.* Contemp. Math., vol. 222. Amer. Math. Soc., Providence, pp. 181–189 (1999). 2, 14

[3] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions.* Acta Math. **149**(1982), no. 1-2, 1–40. 15

[4] E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Michigan Math. J. **5**(1958), 105–126. 1

[5] E. Calabi, *Examples of Bernstein problems for some nonlinear equations*, Proc. Sympos. Pure Math. 15, 223–230, American Mathematical Society, Providence, 1970. 1

[6] R. Harvey and J. Polking, *Removable singularities of solutions of linear partial differential equations.* Acta Math. **125**(1970), 39–56. 19

[7] W. He, *Entire solutions of Donaldson's equation.* Pacific J. Math. **256**(2012), no. 2, 359–363. 2, 17, 18

[8] W. He, On the regularity of the complex Monge–Ampère equations. Proc. Am. Math. Soc. **140**(5)2012, 1719–1727. 2, 14

[9] L. Hörmander, *Notion of convexity.* Birkhäuser Boston, Inc., Boston, MA, 2007, viii+414 pp. 20

[10] K. Jörgens, *Über die Lösungen der Differentialgleichung $rt - s^2 = 1$*, Math. Ann. **127** (1954), 130–134. 1

[11] J. Jost, *Riemannian geometry and geometric analysis*, Universitext Springer, Cham, 2017, xiv+697 pp. 5

[12] N. Q. Le, *Analysis of Monge-Ampère Equations*, Graduate Studies in Mathematics, vol. 240, American Mathematical Society, 2024, xx + 576 pp. 1

[13] S. Myga, *The geometry of C^2 equipped with Warren's metric*, Ann. Polon. Math. **121**(2018), no. 3, 243–249. 8

[14] R. Osserman, *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math. **7** (1957), 1641–1647. 9

[15] A. V. Pogorelov, *The Minkowski multidimensional problem.* Scripta Series in Mathematics 5, V. H. Winston & Sons, Washington, D.C., 1978. Translated from the Russian by Vladimir Oliker. 1

[16] Y. Pan and Y. Zhang, *A residue-type phenomenon and its applications to higer order nonlinear systems of Poisson type.* Journal of Mathematical Analysis and Applications, **495**(2021), no. 2, 30 pp. 2, 3, 4

[17] M. Warren, *A Bernstein result and counterexample for entire solutions to Donaldson's equation.* Proc. Amer. Math. Soc. **144**(2016), no. 7, 2953–2958. 8, 18

[18] J. Wang and W. Wang, *Singular solutions to the complex Monge–Ampère equation.* Math Ann. To appear. 2, 14

pan@pfw.edu,

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA.

zhangyu@pfw.edu,

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499, USA.